Behavior of reflection coefficient

- recall the reflection coefficient at a load is just

$$\rho = \frac{Z - Z_o}{Z + Z_o} = \frac{\dfrac{R + j\cdot X}{Z_o} - 1}{\dfrac{R + j\cdot X}{Z_o} + 1} = \frac{(r + j\cdot x) - 1}{(r + j\cdot x) + 1}$$

- how does rho behave as “load” varies?
  - $Z_{load} = 0$ (short) ⇨ rho = -1
  - $Z_{load} = \infty$ (open) ⇨ rho = +1
  - $Z_{load} =$ purely imaginary number ⇨ $|\rho| = 1$

$$\rho = \frac{(j\cdot x) - 1}{(j\cdot x) + 1} = \frac{-(x)^2 + 1 - 2\cdot(j\cdot x)}{x^2 + 1}$$

$$|\rho| = \sqrt{\frac{-(x)^2 + 1 - 2\cdot(j\cdot x)}{x^2+1}\cdot\frac{-(x)^2 + 1 + 2\cdot(j\cdot x)}{x^2+1}} = \frac{\sqrt{x^4 - x^2 - 2jx^3 - x^2 + 1 + 2jx + 2jx^3 - 2jx + 4x^2}}{x^2+1}$$

$$= \frac{\sqrt{x^4 + 1 + 2x^2}}{x^2 + 1} = \frac{\sqrt{(x^2+1)^2}}{x^2 + 1} = 1$$

Relationships between rho and Z

- **note relation between rho and the "normalized impedance"**

$$\frac{1+\rho}{1-\rho} = \frac{1+\dfrac{[(r+j\cdot x)-1]}{[(r+j\cdot x)+1]}}{1-\dfrac{[(r+j\cdot x)-1]}{[(r+j\cdot x)+1]}}\cdot\frac{[(r+j\cdot x)+1]}{[(r+j\cdot x)+1]} = \frac{[(r+j\cdot x)+1]+[(r+j\cdot x)-1]}{[(r+j\cdot x)+1]-[(r+j\cdot x)-1]} = \frac{2r+j\cdot 2x}{2}$$

$$\Rightarrow \quad \frac{1+\rho}{1-\rho} = r + j\cdot x$$

- **consider rho in complex plane** $\rho = u + j\cdot v$

$$r + j\cdot x = \frac{1+\rho}{1-\rho} = \frac{1+(u+j\cdot v)}{1-(u+j\cdot v)} = \frac{(1+u)+j\cdot v}{(1-u)-j\cdot v}\cdot\frac{(1-u)+j\cdot v}{(1-u)+j\cdot v} = \frac{1-u^2-v^2+2\cdot j\cdot v}{(1-u)^2+v^2}$$

- **so the relationship between the normalized impedance and the real and imaginary parts of rho is**

$$r + j\cdot x = \frac{1-\left(u^2+v^2\right)}{(1-u)^2+v^2} + j\cdot\frac{2\cdot v}{(1-u)^2+v^2}$$

Relationships between rho and Z

$$r + j \cdot x = \frac{1 - \left(u^2 + v^2\right)}{(1-u)^2 + v^2} + j \cdot \frac{2 \cdot v}{(1-u)^2 + v^2} \qquad \rho = u + j \cdot v$$

- **from the real part:**

$$r \cdot \left[(1-u)^2 + v^2\right] = 1 - \left(u^2 + v^2\right)$$

$$r \cdot \left[1 - 2u + u^2 + v^2\right] = 1 - \left(u^2 + v^2\right)$$

$$u^2 \cdot (r+1) - 2ur + v^2 \cdot (r+1) = 1 - r$$

$$u^2 - 2u\frac{r}{r+1} + v^2 = \frac{1-r}{r+1}$$

$$\left[u^2 - 2u\frac{r}{r+1} + \left(\frac{r}{r+1}\right)^2\right] + v^2 = \frac{1-r}{r+1} + \left(\frac{r}{r+1}\right)^2$$

$$\left[u - \frac{r}{r+1}\right]^2 + v^2 = \frac{(1-r)\cdot(1+r) + r^2}{(r+1)^2}$$

$$\left[u - \frac{r}{r+1}\right]^2 + v^2 = \frac{1 - r^2 + r^2}{(r+1)^2} \Rightarrow \boxed{\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1}{(1+r)^2}}$$

Plot of reflection coefficient in complex plane

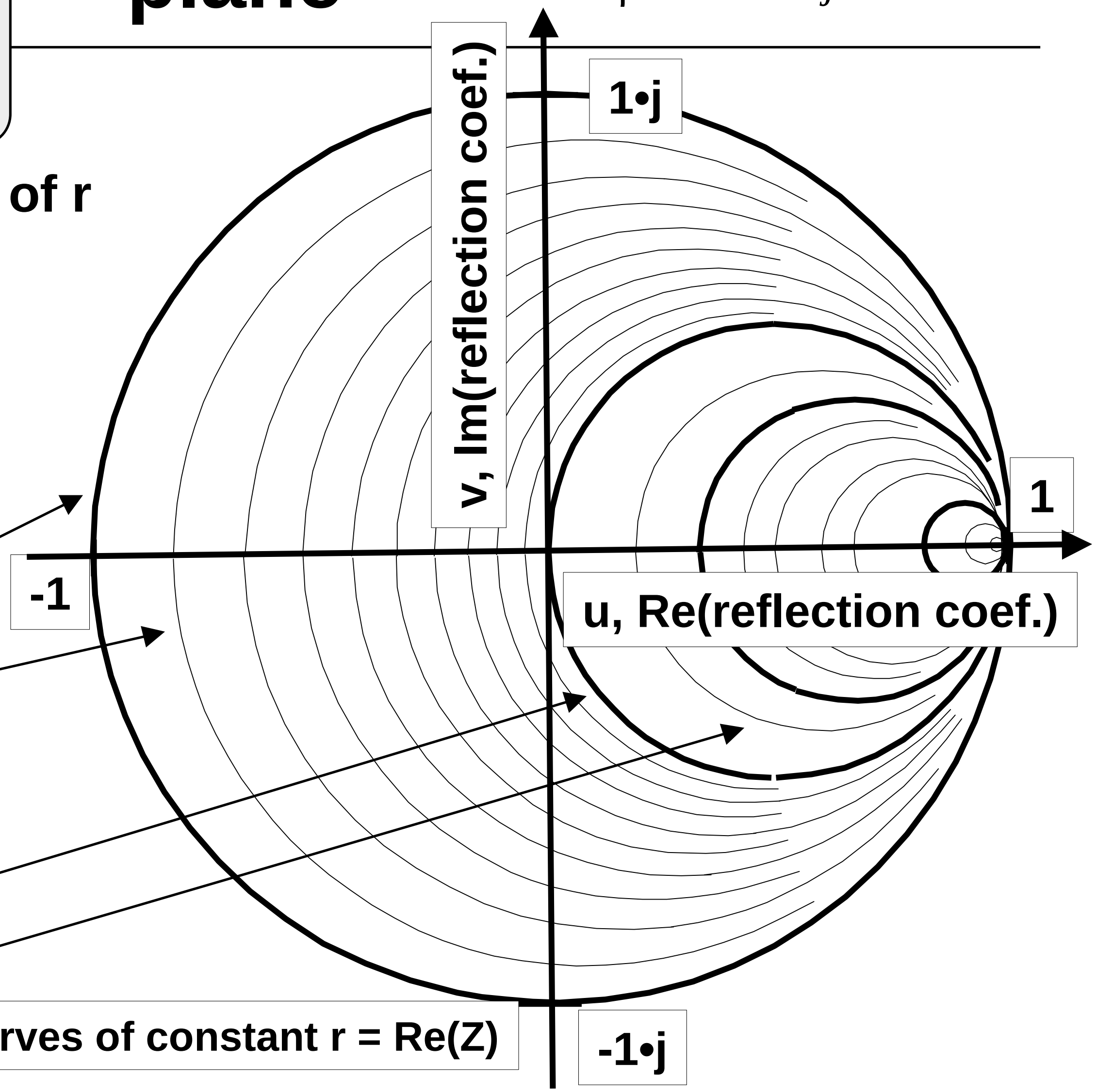
$$\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1}{(1+r)^2}$$

$$\rho = u + j \cdot v$$

- plot rho as a function of r
  - these are circles!
    - center $\left(\frac{r}{1+r},\ 0\right)$
    - radius $\frac{1}{1+r}$
  - r = 0

    $$(u-0)^2 + v^2 = \left(\frac{1}{1}\right)^2$$
  - 0 < r < 1
  - r = 1

    $$\left(u - \frac{1}{2}\right)^2 + v^2 = \left(\frac{1}{2}\right)^2$$
  - r > 1



Plot of reflection coefficient

$$r + j\cdot x = \frac{1-\left(u^2+v^2\right)}{(1-u)^2+v^2} + j\cdot\frac{2\cdot v}{(1-u)^2+v^2} \qquad \rho = u + j\cdot v$$

- **from the Im part:**

$$(u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$

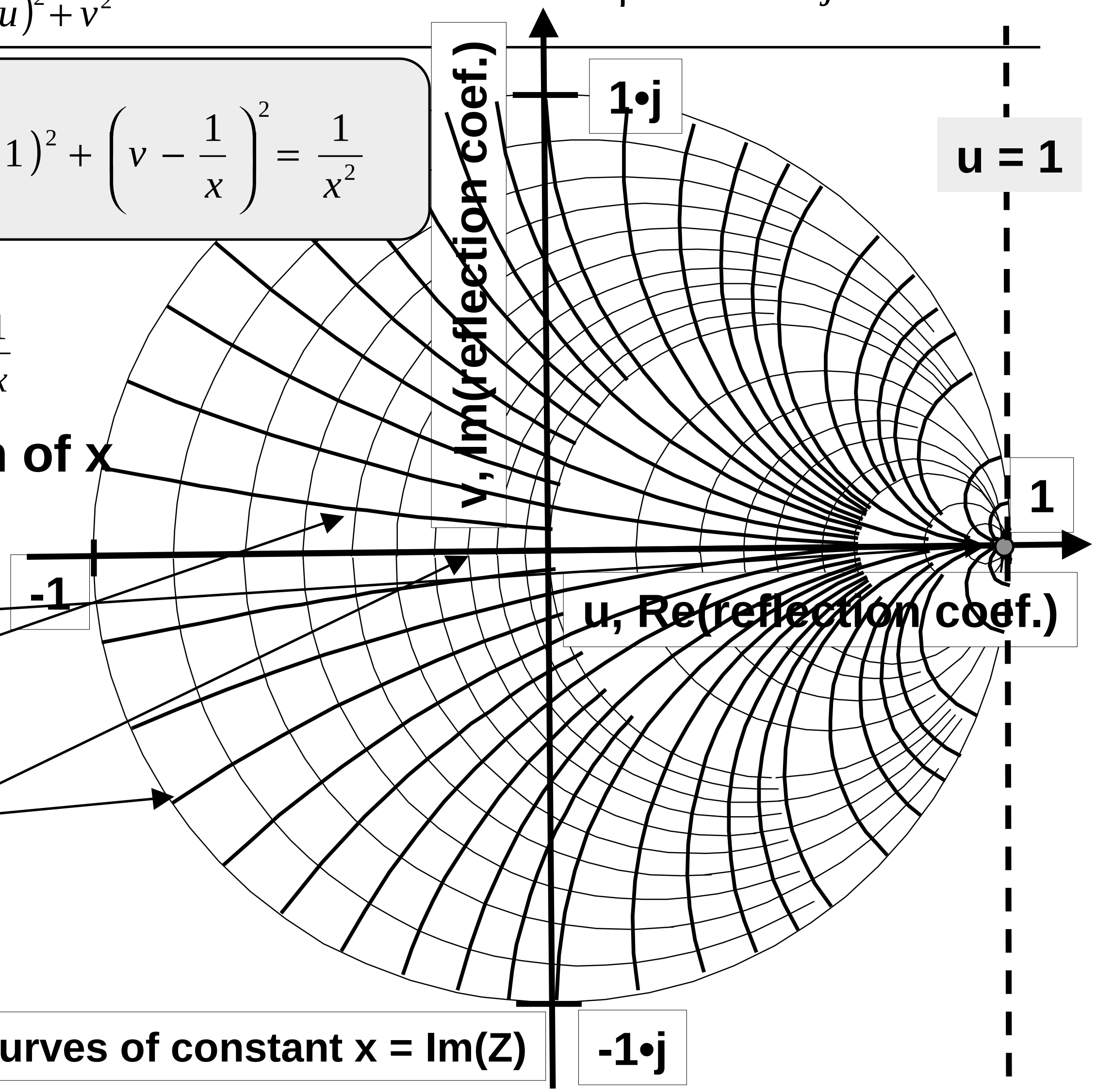
  - **these are also circles!**
    - **center** $\left(1\ ,\ \frac{1}{x}\right)$
    - **radius** $\frac{1}{x}$

- **plot rho as a function of x**
  - **x = ± ∞**

$$(u-1)^2 + \left(v - \frac{1}{\pm\infty}\right)^2 = \frac{1}{\infty}$$

  - **x > 0**
  - **x < 0**
  - **x = 0**

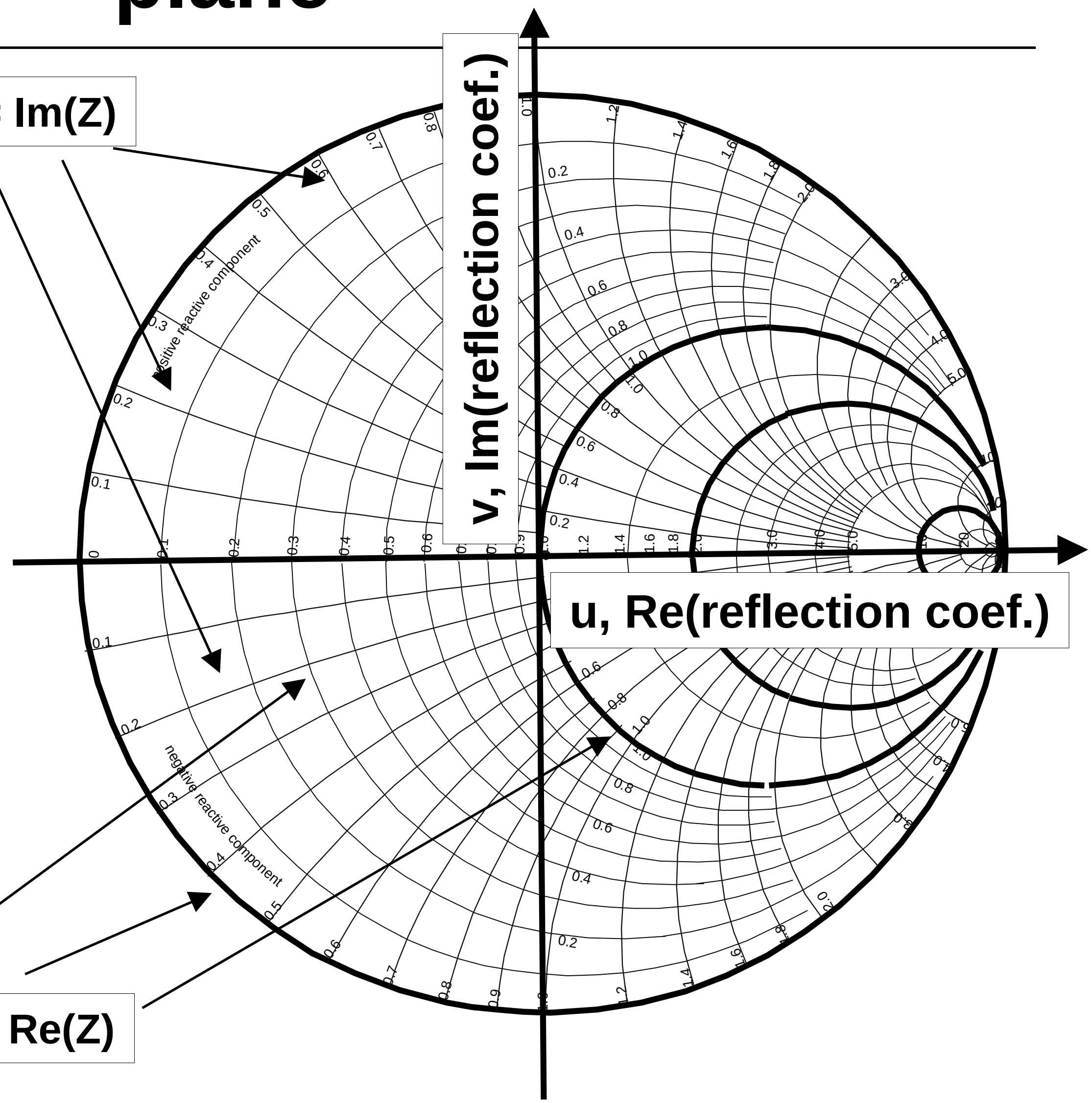
$$(u-1)^2 + \left(v - \frac{1}{0}\right)^2 = \frac{1}{0^2}$$



Plot of reflection coefficient in complex plane

$$\rho = \frac{(r + j \cdot x) - 1}{(r + j \cdot x) + 1}$$

- **the Smith Chart is a plot of the reflection coefficient in the complex plane, with contours of constant load resistance and load reactance superimposed**



Notes on rho

- **for any non-negative value of Re(Z) (i.e., r ≥ 0) and any value of Im(Z) rho falls on or within the unit circle in the complex rho plane**
- **recalling that the reflection coefficient for a lossless transmission line of length *l* terminated by impedance Z_L is**

$$\rho(-l) = \frac{Z_L - Z_o}{Z_o + Z_L} \cdot e^{-j2\beta l}$$

  - **so in the complex rho plane you trace out a circle of constant radius as *l* varies**
    - **radius of circle is just** $\left|\frac{Z_L - Z_o}{Z_o + Z_L}\right|$
    - **recall the input impedance was**

$$Z_{in} = Z_o \cdot \frac{Z_L - j \cdot Z_o \cdot \tan(\beta \cdot l)}{Z_o + j \cdot Z_L \cdot \tan(\beta \cdot l)}$$

    - **you can read the values right off the Smith chart!**

Plot of reflection coefficient in complex plane

$$\rho = \frac{(r + j \cdot x) - 1}{(r + j \cdot x) + 1}$$