Compact Equivalent Circuit Models for the Skin Effect

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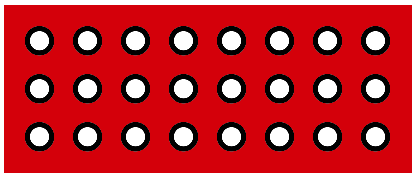
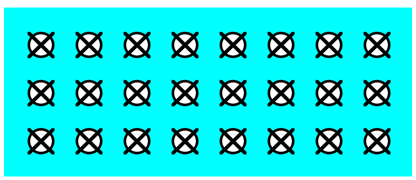
Origin of frequency dependencies in transmission line series impedance

Low frequencies

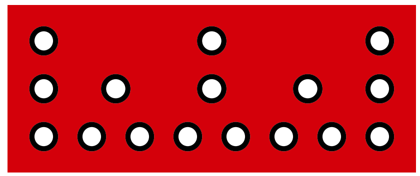


Uniform Current: dc

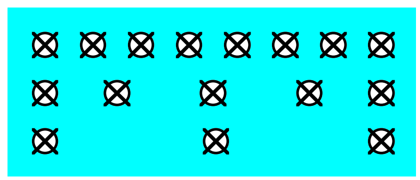


Resistance: R_{dc}
Inductance: uniform current distribution

Mid frequencies

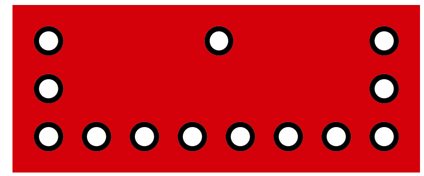


Non-Uniform: proximity

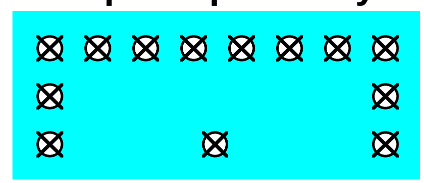


Resistance: increases
Inductance: decreases

High frequencies



Non-Uniform: skin depth & proximity

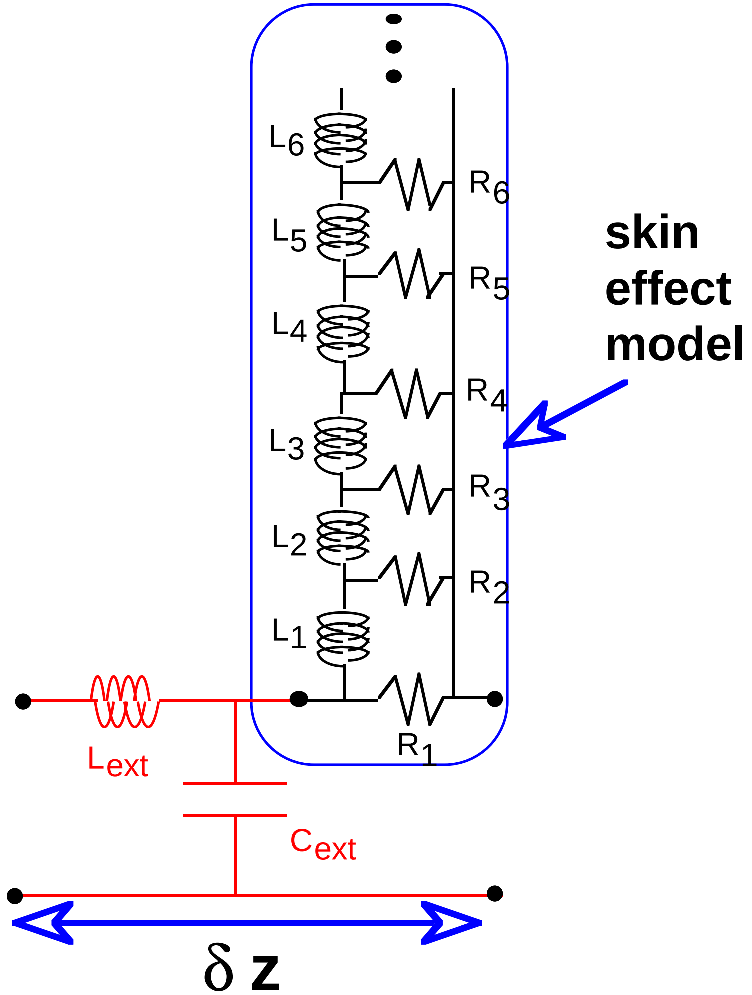


Resistance: increases
Inductance: constant, infinite conductivity (high frequency) limit

- **can frequency independent ladder circuits be synthesized to accurately model frequency dependent series impedance of line?**

R-L ladder circuits for the skin effect

- use of R-L ladders is classical technique
  - e.g., H. A. Wheeler, “Formulas for the skin-effect,” Proceedings of the Institute of Radio Engineers, vol. 30, pp. 412-424, 1942.
- essentially an application of transverse resonance
- lumping based on uniform step size tends to generate large ladders



Non-Uniform "step" size for compact ladders

- for lossy transmission lines and bandwidth limited signals, can use increasingly long step size as propagate along line
  - line acts like a low pass filter, so as you propagate along the line the effective bandwidth decreases, allowing longer steps
- for a skin effect equivalent circuit of a circular wire, Yen et al. proposed use of steps such that the resistance ratio RR from one step to the next is a constant

$$R_i / R_{i+1} = RR \quad \longrightarrow \quad R_i = \frac{1}{\sigma \pi r^2} \cdot \sum_{j=0}^{M-1} (RR)^{M-j-i}$$

for an M-deep ladder this leads to

radii of rings:

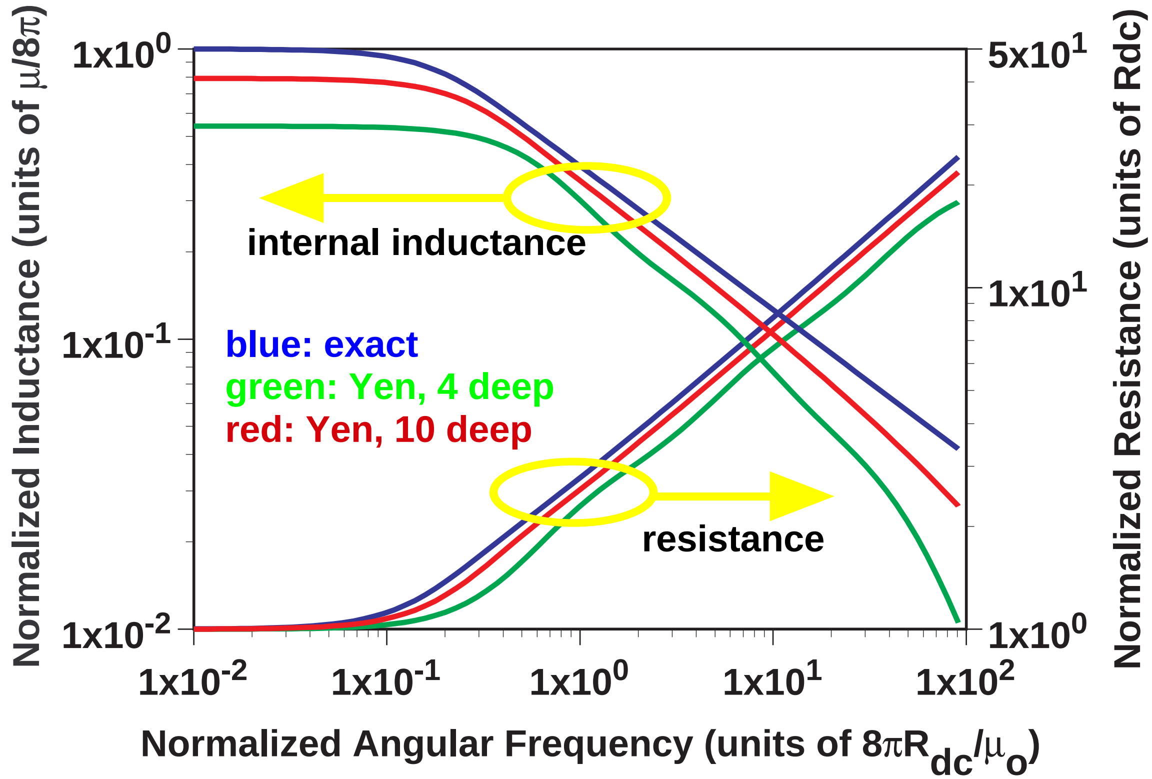
$$r_i = r \cdot \sqrt{\sum_{j=i+1}^{M} \left(\sum_{n=1}^{M} (RR)^{M-j-n+1} \right)^{-1}}$$

inductances:

$$L_i = \frac{\mu \cdot (r_{i-1} - r_i)}{2\pi \cdot r_i}$$

[C.-S. Yen, Z. Fazarinc, and R. L. Wheeler, “Time-Domain Skin-Effect Model for Transient Analysis of Lossy Transmission Lines,” Proceedings of the IEEE, vol. 70, pp. 750-757, 1982]

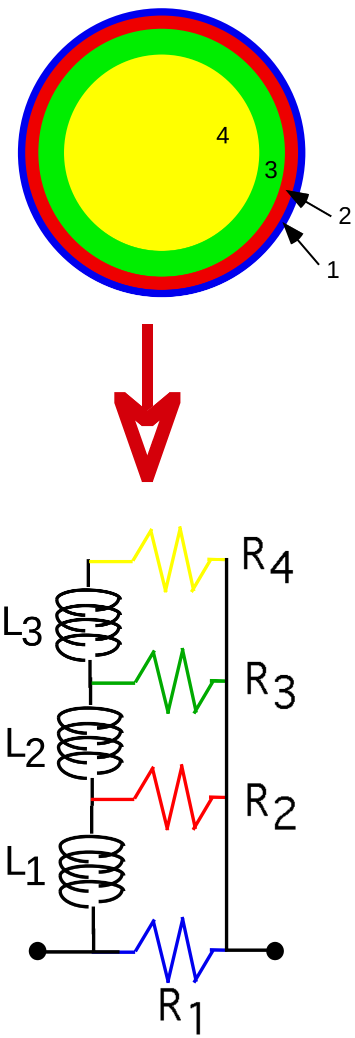
Yen's results for a single circular wire



- selection of ladder length and RR determines accuracy:
  - $m = 4$ (i.e., 4 resistors, 3 inductors), minimum error occurs for RR = 2.31
  - $m = 10$, minimum error for RR = 1.37

"Compact" ladders

- problem: Yen's approach tends to underestimate both resistance and inductance
- can a "short" ladder produce a good approximation?
  - "de-couple" resistance and inductance in a 4-long ladder
  - each shell such that
    - $R_i / R_{i+1} = \mathit{RR}$, a constant (> 1)
      - $R_2 = \mathit{RR}\ R_1$, $R_3 = \mathit{RR}^2\ R_1$, $R_4 = \mathit{RR}^3\ R_1$
    - $L_i / L_{i+1} = \mathit{LL}$, a constant (< 1)
      - $L_2 = \mathit{LL}\ L_1$, $L_3 = \mathit{LL}^2\ L_1$



Fitting parameters for 4-long ladder

- "unknowns" constrained by asymptotic behavior at low frequency
  - given the dc resistance R_{dc}, then R_1 and RR are related by:

$$(RR)^3 + (RR)^2 + RR + \left(1 - \frac{R_1}{R_{dc}}\right) = 0$$

  - given the low frequency internal inductance $L_{lf}^{internal}$, then L_1 and LL are related by:

$$\left(\frac{1}{LL}\right)^2 + \left(1 + \frac{1}{RR}\right)^2 \frac{1}{LL} + \left(\left[\frac{1}{RR}\right]^2 + \frac{1}{RR} + 1\right)^2 - \frac{L_{lf}^{internal}}{L_1}\left(\left[1 + \frac{1}{RR}\right]\left[\left\{\frac{1}{RR}\right\}^2 + 1\right]\right)^2 = 0$$

- only "free" fitting parameters are R_1 and L_1 (or equivalently, RR and LL)
  - R_1 and L_1 tend to dominate the high frequency response

Best fit for single circular wire

- "universal" fit possible over specified bandwidth (dc to ω_{max})
- scales in terms of radius compared to minimum skin depth (that occurs at highest frequency)

$$\delta_{\max} = \sqrt{\frac{2}{\omega_{\mathbf{max}} \mu_o \sigma}}$$

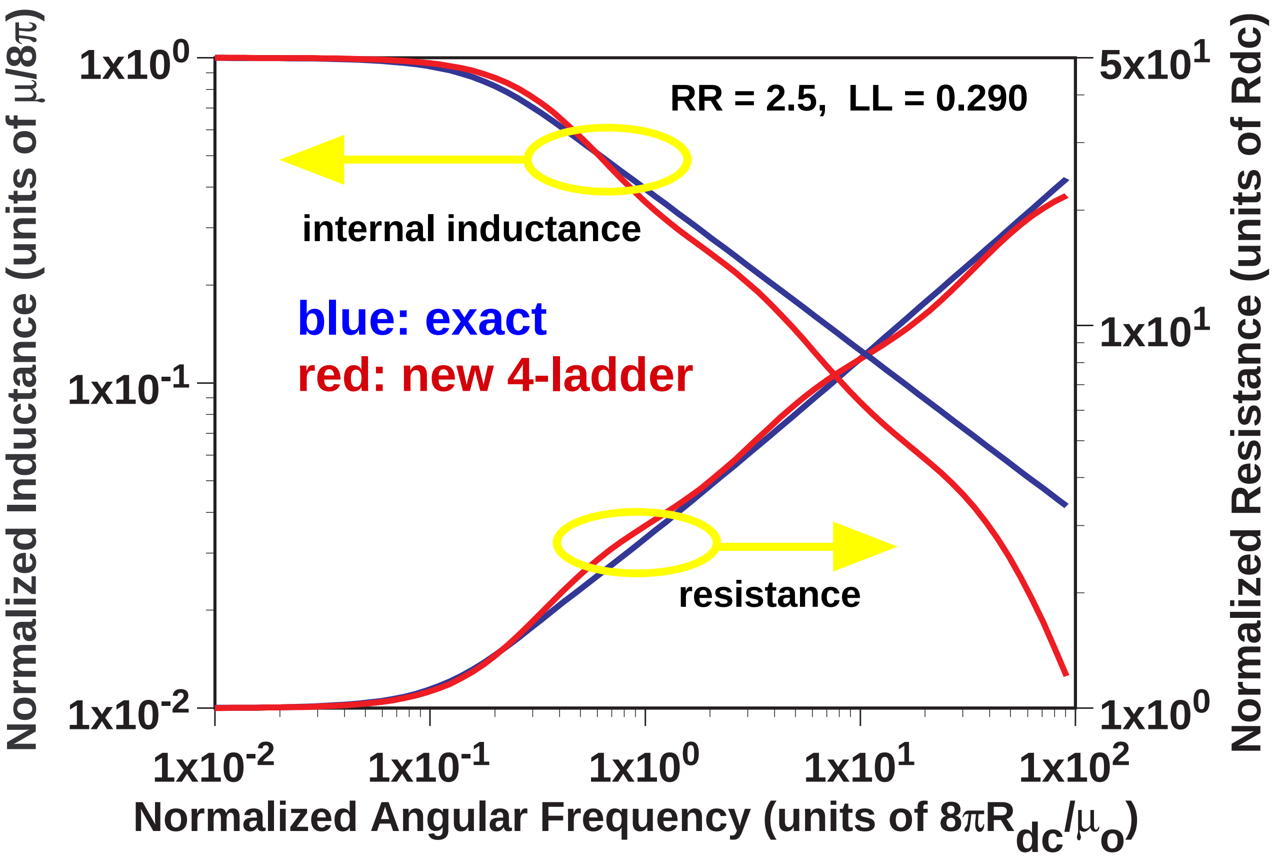
R_1 (and hence RR):

$$\frac{R_1}{R_{dc}} = 0.53 \frac{\text{wire radius}}{\delta_{\mathbf{max}}}$$

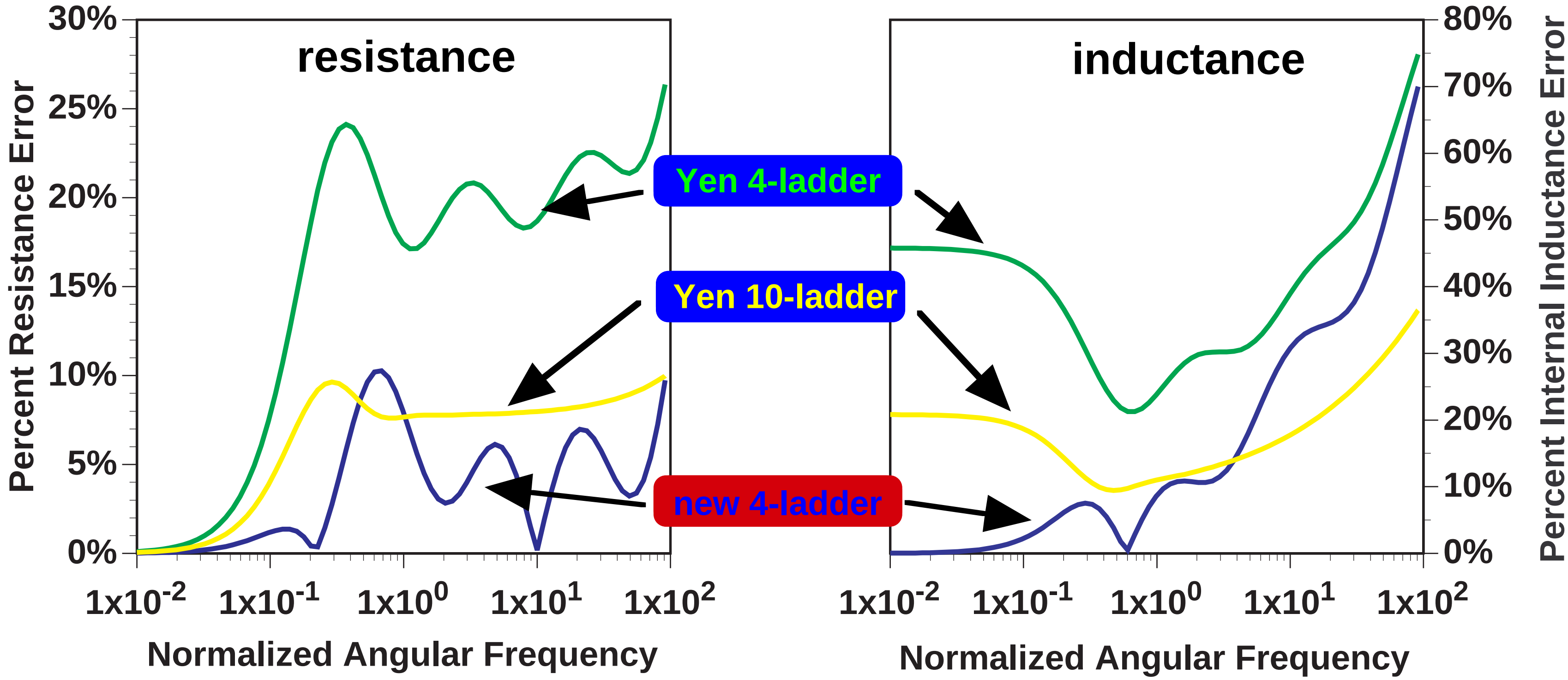
L_1 (and hence LL):

$$\frac{L_{lf}^{internal}}{L_1} = 0.315 \cdot \frac{R_1}{R_{dc}}$$

Results for single circular wire

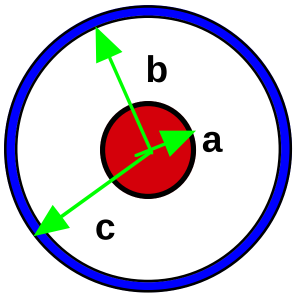


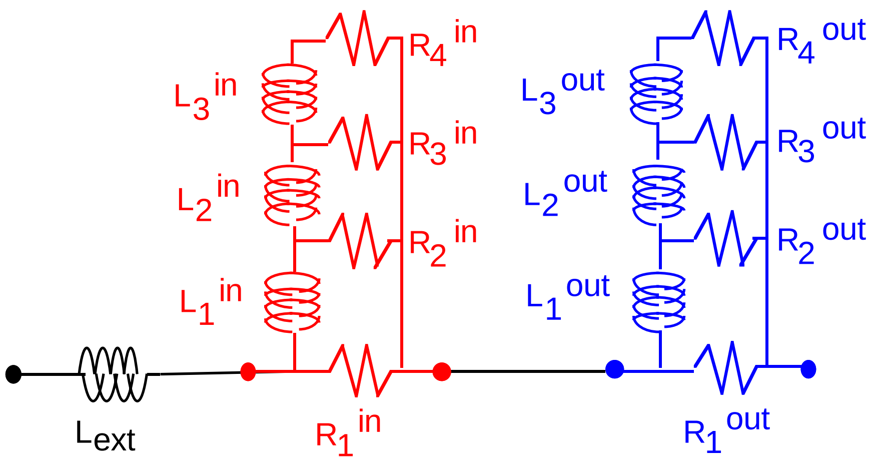
Errors for single circular wire

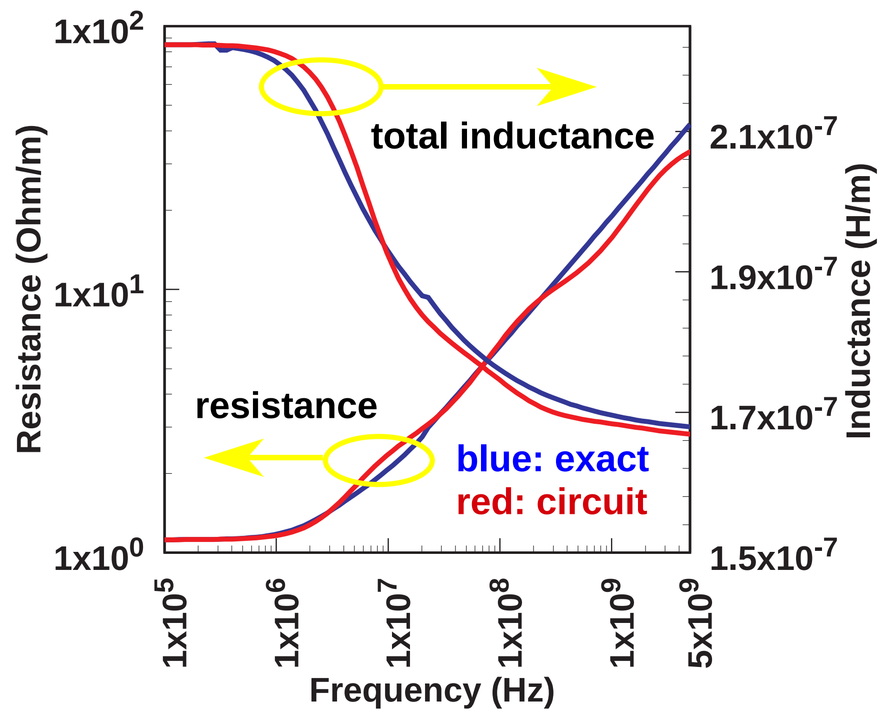


- excellent fit possible over wide range of frequencies, from low to high frequency
- shorter ladders (three of less) give much larger errors
- longer ladders improve accuracy very slowly

Results for coaxial cable







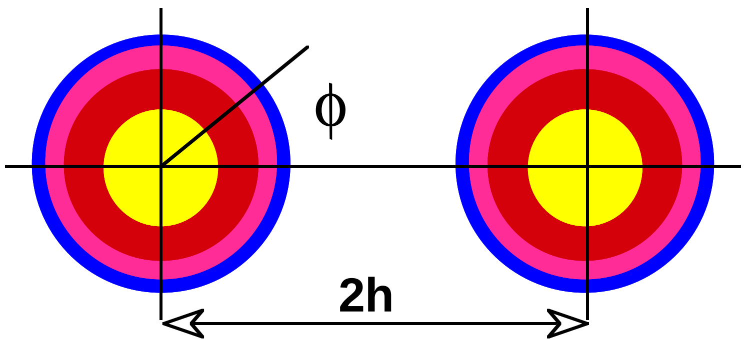
- **can account for both inner (signal) and outer (shield) conductors**

example:
inner radius a = 0.1 mm
shield radius b = 0.23 mm
shield thickness 0.02 mm
f_{max} = 5 GHz

Inclusion of proximity effects

- for transmission lines with "non-circular" geometry must also account for proximity effects
- use high frequency behavior to estimate current division over surfaces of conductors
  - subdivide external inductance (L_{ext}) to force current redistribution

Twin lead with proximity effect

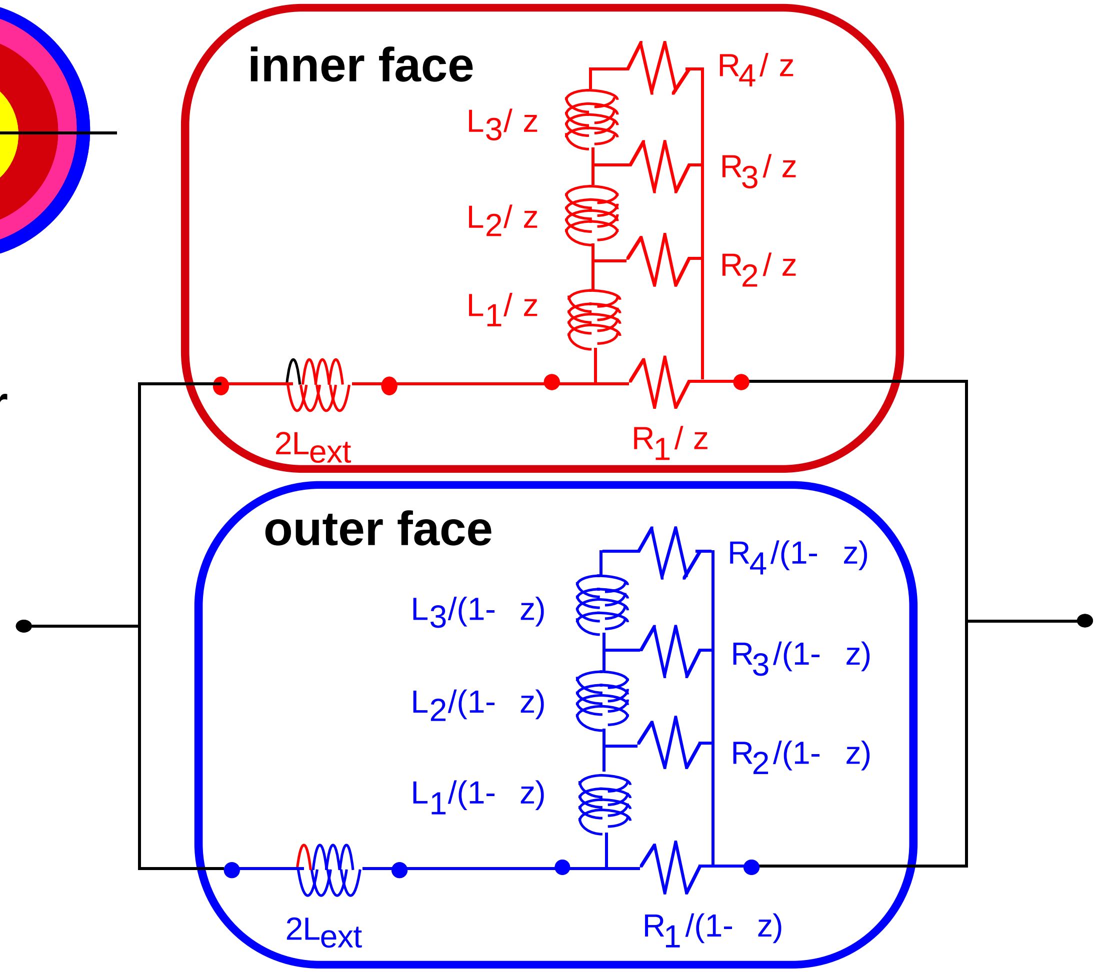


- **more flux coupling at inner faces**
  - **quarter from angle ϕ**

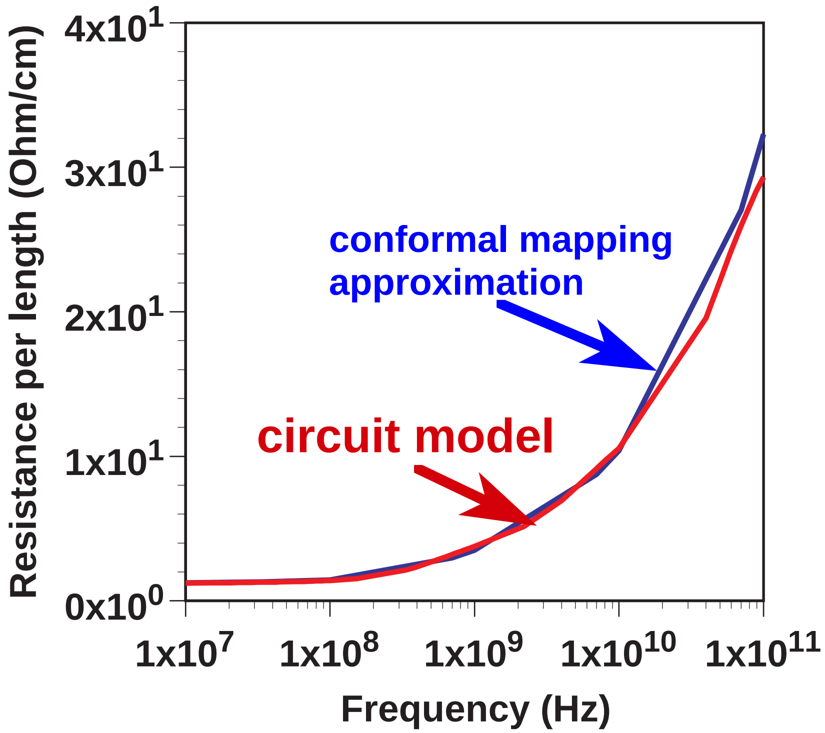
$$\sin(\phi) = \sqrt{1 - (r/h)^2}$$

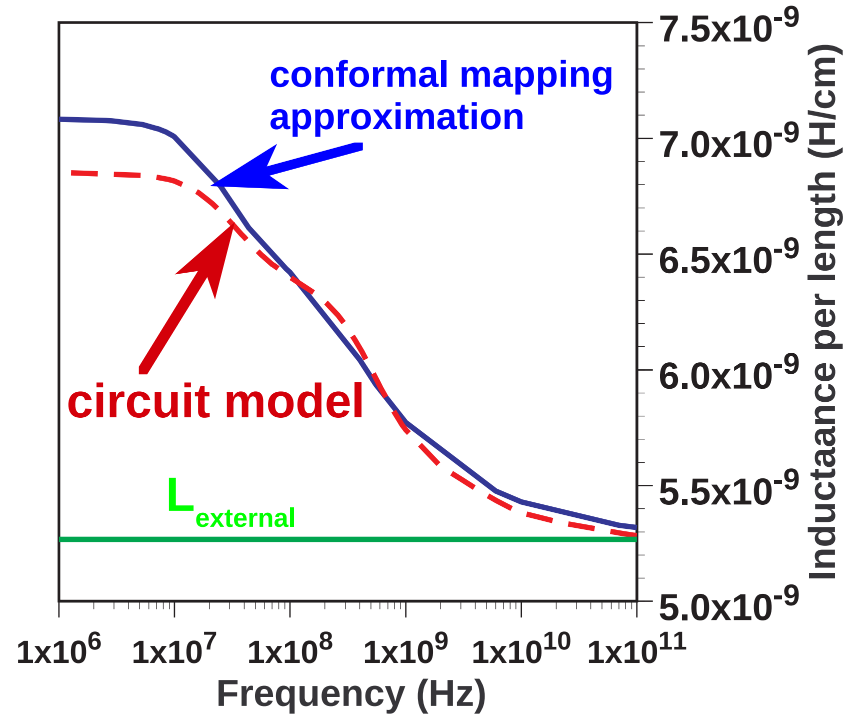
- **two branches required**
- **weight skin effect by ζ**

$$\zeta = \phi / \pi$$



Results for closely coupled twin lead

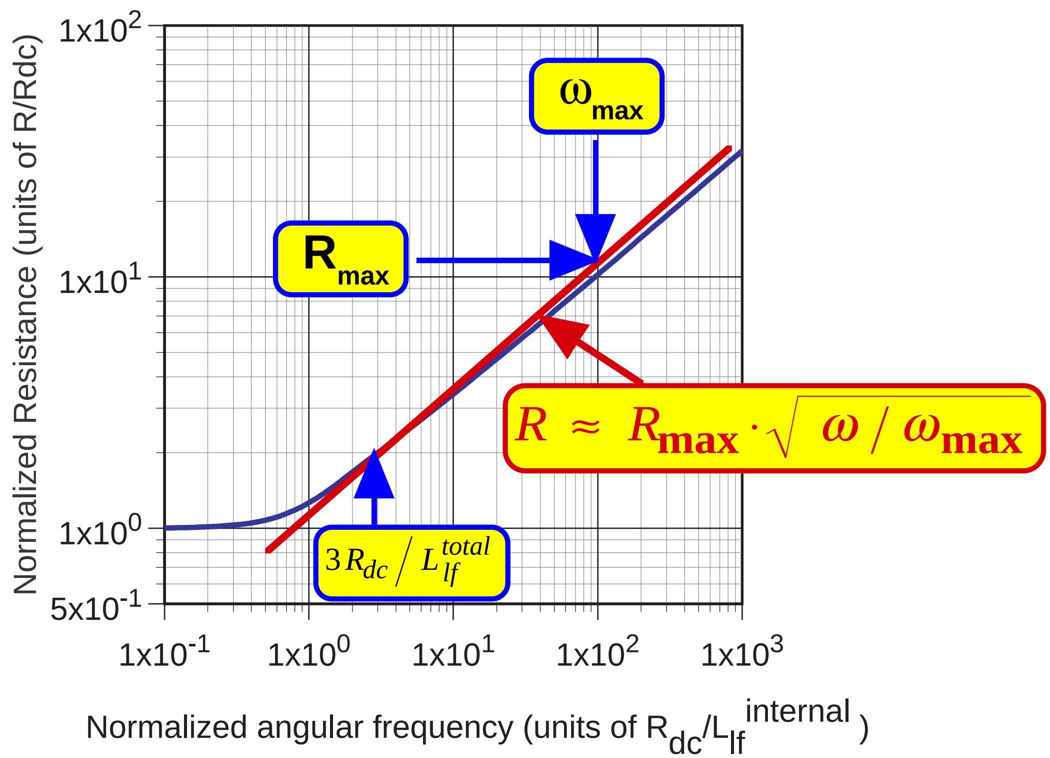




- example for 1 mil diameter Al wires on 2 mil centers
  - $\phi = 60^\circ$

Generalized circuit generation

- observation:
  - regardless of geometry of transmission line, for frequencies greater than about $3R_{dc}/L_{lf}$, resistance increases as $\sqrt{\omega}$
- can force single 4-long ladder circuit response to pass through a given high frequency point with $\sqrt{\omega}$ dependence
  - should work for noncircular geometries, even with strong proximity effects



General fitting procedure

- **Objective: force high frequency circuit response to pass through R_{max} at ω_{max}**
  - **high frequency asymptotic behavior of 4-ladder is**

$$Z_{hf}^{circuit} \approx \frac{R_1 \left(R_1 \cdot RR^{-1} + j\,\omega\, L_1 \right)}{R_1 \cdot \left(1 + RR^{-1} \right) + j\,\omega\, L_1} \qquad \textbf{(eq. 1)}$$

- **for a given choice of RR, from dc requirements find R_1:**

$$R_1 = R_{dc} \left(RR^3 + RR^2 + RR + 1 \right) \qquad \textbf{(eq. 2)}$$

- **require that $R_{circuit} = R_{max}$ at ω_{max}:**

$$R_{\mathbf{max}} = R_1 \frac{RR^{-1} \cdot \left(1 + RR^{-1} \right) + \left(\dfrac{\omega_{\mathbf{max}}\, L_1}{R_1} \right)^2}{\left(1 + RR^{-1} \right)^2 + \left(\dfrac{\omega_{\mathbf{max}}\, L_1}{R_1} \right)^2} \qquad \textbf{(eq. 3)}$$

Generalized fitting procedure

•so L_1 is given by:

$$L_1 = \frac{R_{dc}\left(RR^3 + RR^2 + RR + 1\right)(1 + 1/RR)}{\omega_{\mathbf{max}}} \sqrt{\frac{R_{\mathbf{max}} - R_{dc}\left(1 + RR^2\right)}{R_{dc}\left(RR^3 + RR^2 + RR + 1\right) - R_{\mathbf{max}}}}$$

(eq. 4)

•and finally by LL is found using the dc requirement:

$$LL^{-2} + LL^{-1}\left(RR^{-1} + 1\right)^2 + \left(RR^{-2} + RR^{-1} + 1\right)^2 - \frac{L_{lf}^{internal}}{L_1}\left(RR^{-3} + RR^{-2} + RR^{-1} + 1\right)^2 = 0$$

(eq. 5)

where

$$L_{lf}^{internal} = L_{lf}^{total} - L_{hf}^{external}$$

(eq. 6)

Summary of procedure

- find low and high frequency behavior
  - R_{dc}, L_{lf}^{total}, $L_{hf}^{external}$, R_{max} at single high frequency ω_{max}
  - could be determined by either calculation or measurement
- iterate to find optimum RR
  - since $R_1 > R_{max}$, RR is bounded below such that:

$$\frac{R_{\mathbf{max}}}{R_{dc}} \leq (RR)^3 + (RR)^2 + RR + 1 \qquad \text{(eq. 7)}$$

  - constraint on real value for L_1 produces an upper bound

$$RR^2 + 1 < \frac{R_{\mathbf{max}}}{R_{dc}}$$

  - hence RR must satisfy the inequality

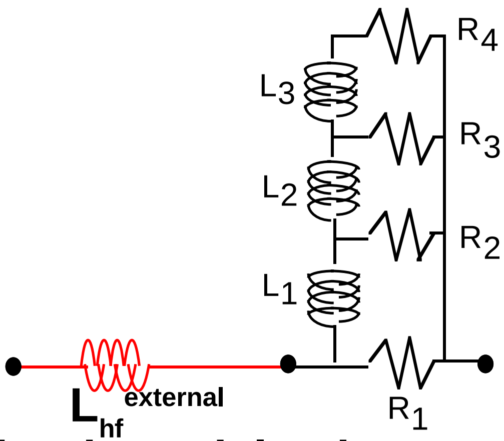
$$1 + RR^2 < \frac{R_{\mathbf{max}}}{R_{dc}} < RR^3 + RR^2 + RR + 1$$

Summary of procedure

- **start with RR at lower bound (eq. 7)**
- **calculate R_1 from eq. 2**
- **calculate L_1 from eq. 4**
- **calculate LL from eq. 5**
- **use resulting 4-ladder to calculate circuit response over interval from $3R_{dc}/L_{lf}$ to ω_{max} (interval over which $\sqrt{\omega}$ behavior holds)**
  - **find error between circuit and assumed response** $R \approx R_{\mathbf{max}} \cdot \sqrt{\omega / \omega_{\mathbf{max}}}$
- **increment RR, find new error**
  - **continue until error is minimized**

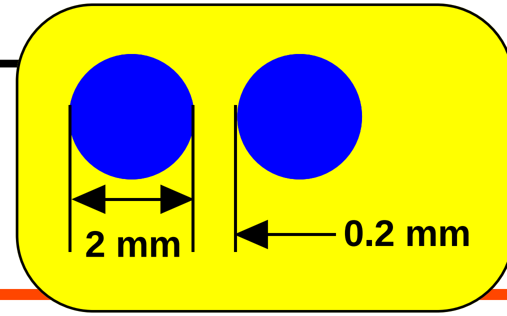
Examples for generalized fitting

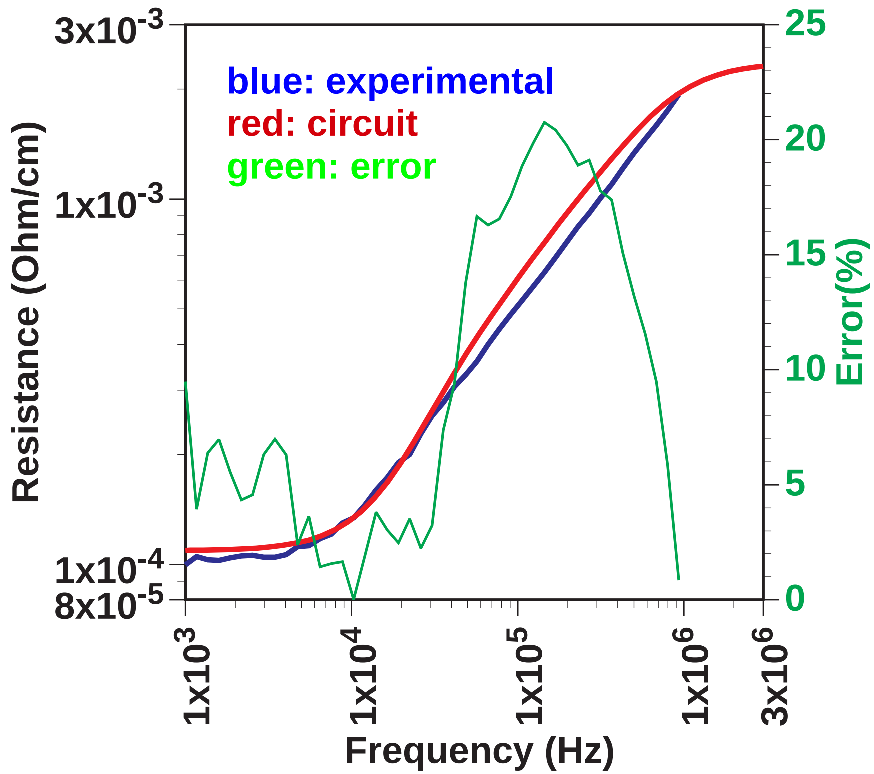
- series equivalent per unit length circuit for transmission line is

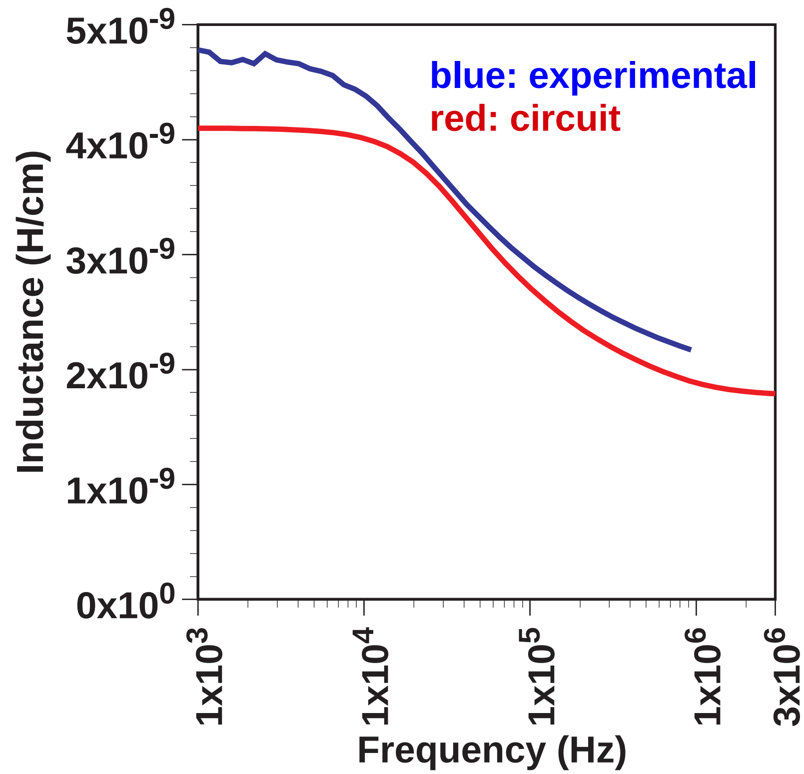


- verification of circuit model using:
  - experimental results for closely coupled twin lead
    - experimentally measured resistance and inductance data
    - fit to experimental resistance, calculation for L_{lf}^{total}, $L_{hf}^{external}$
  - full volume filament calculations for wide range of rectangular geometries
    - parallel thick plates
    - coplanar lines
    - parallel square bars

Closely coupled twin lead



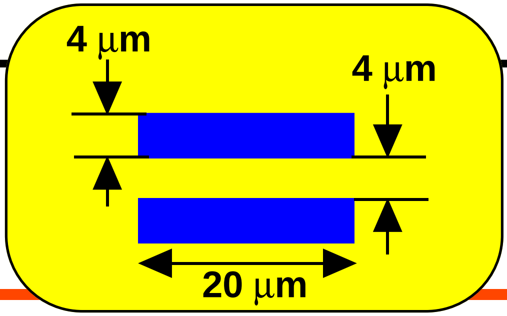


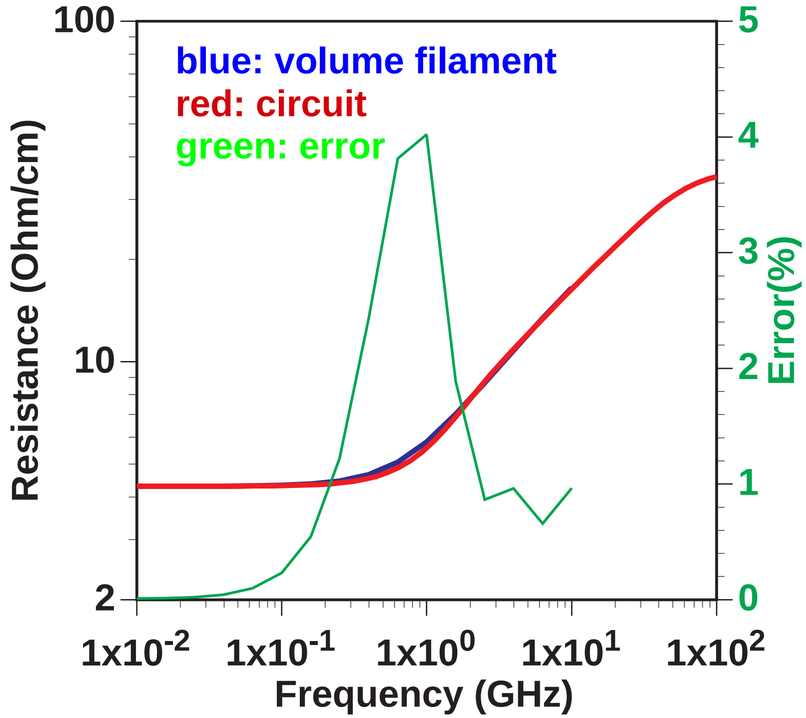


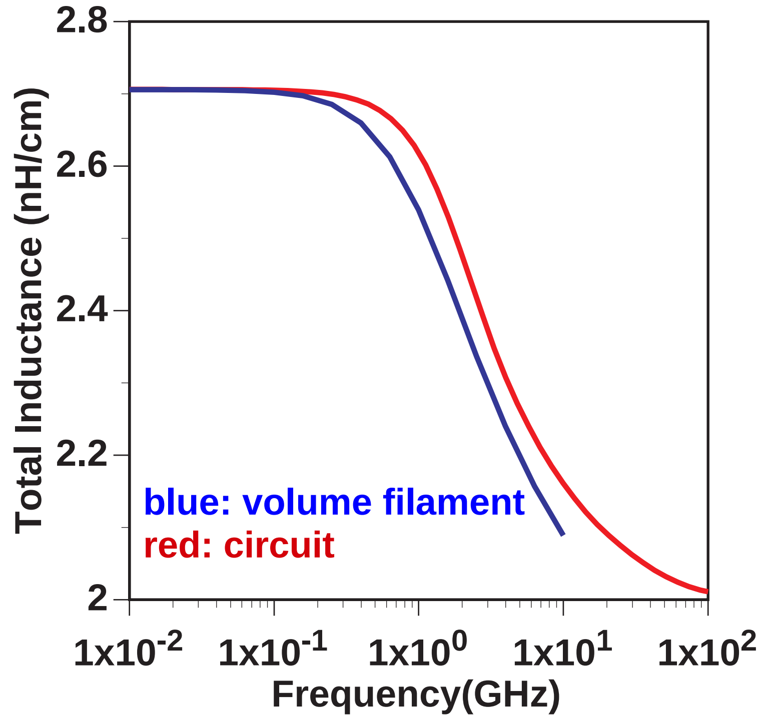
- R_{dc} = 0.01 Ω/m , L_{lf}^{total} = 4.1 x 10^{-7} H/m , $L_{hf}^{external}$ = 1.77 x 10^{-7} H/m
- f_{max} = 9.33 x 10^{5} Hz , R_{max} = 0.193 Ω/m

→ **RR = 2.34 , LL = 0.782**

Parallel thick plates



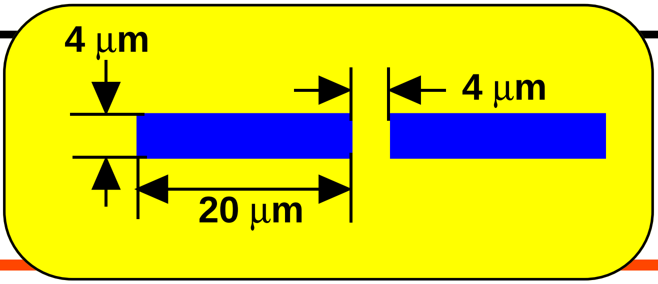


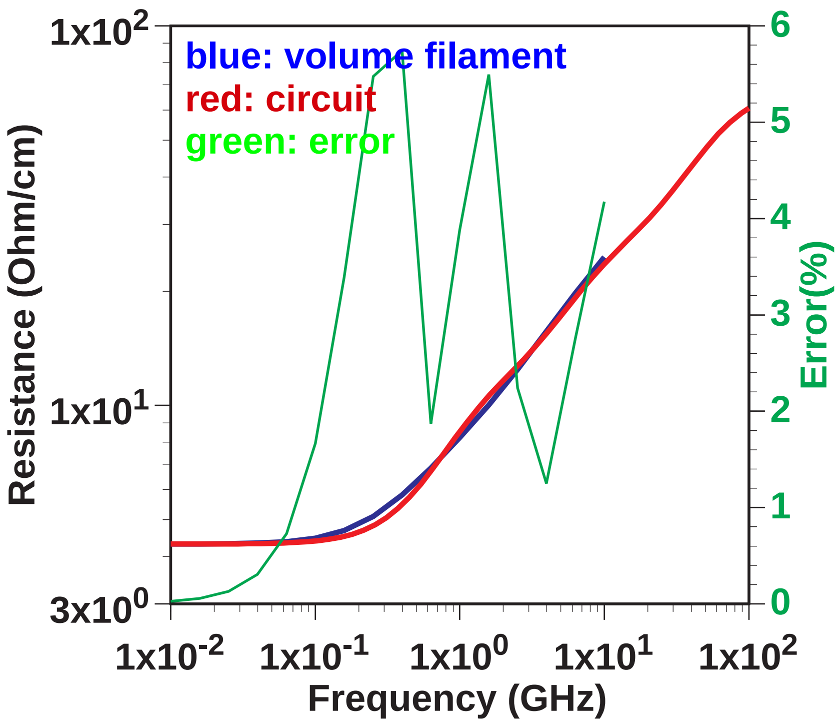


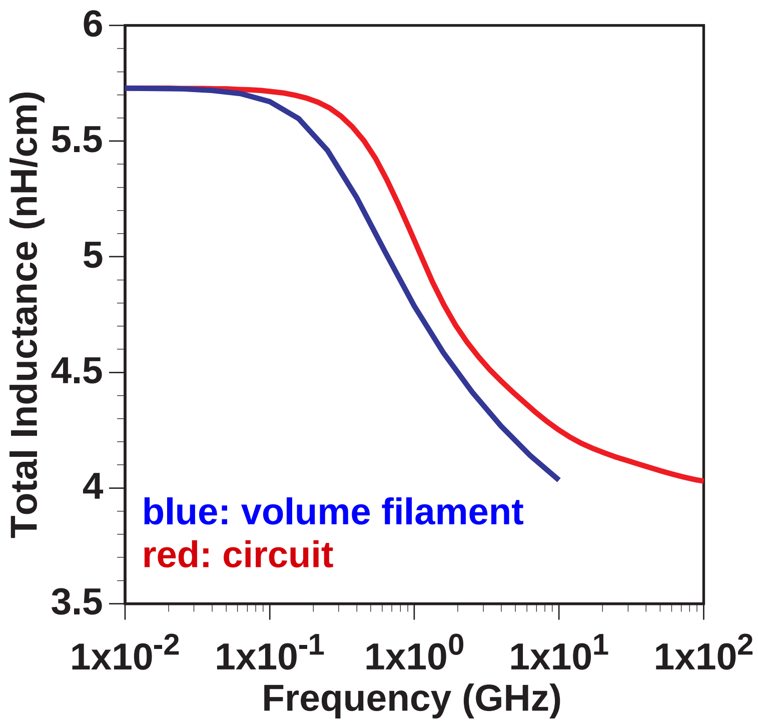
- R_{dc} = 431 Ω/m , L_{lf}^{total} = 2.7 x 10^{-7} H/m , $L_{hf}^{external}$ = 2 x 10^{-7} H/m
- f_{max} = 1 x 10^{10} Hz , R_{max} = 1650 Ω/m

→ **RR = 1.54 , LL = 0.523**

Coplanar lines



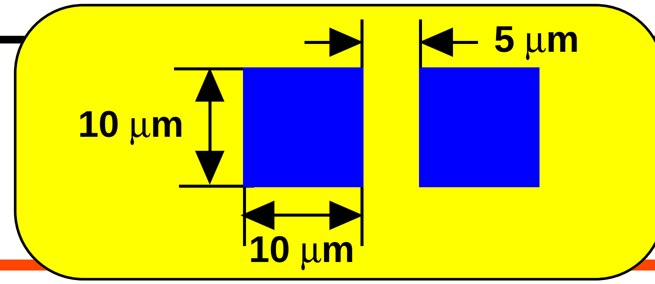


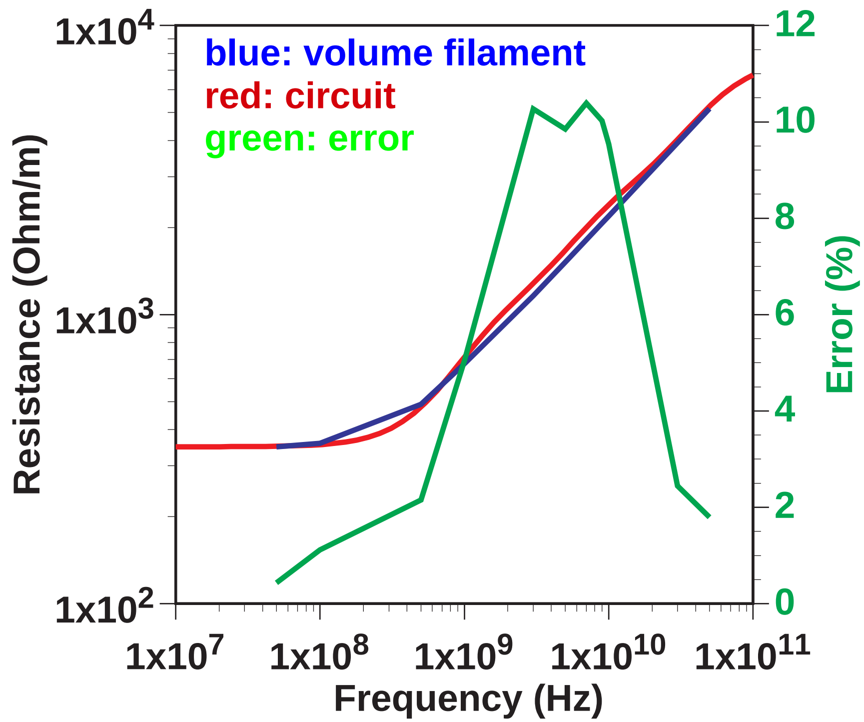


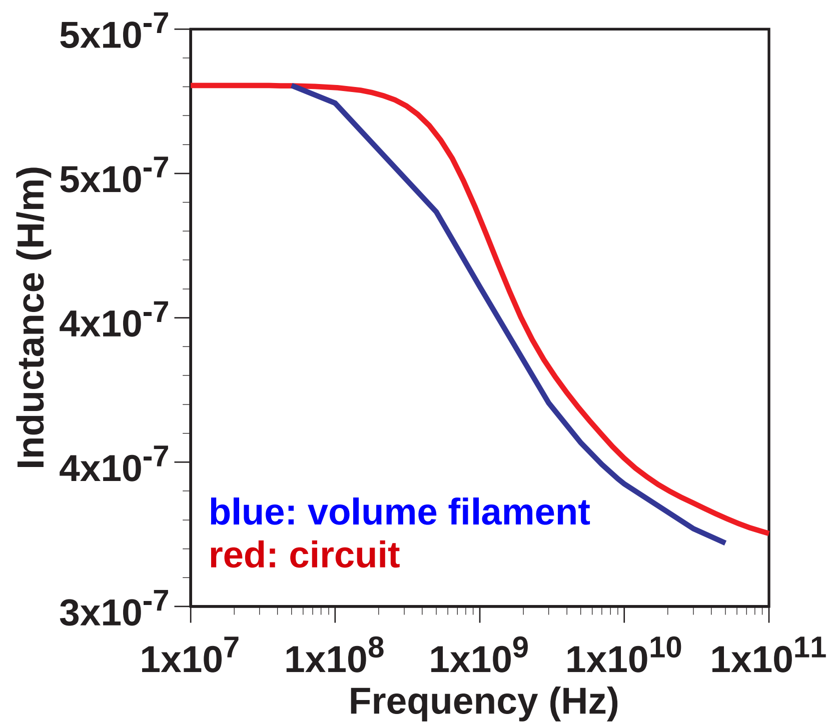
- $R_{dc} = 431\ \Omega/m$, $L_{lf}^{total} = 5.7 \times 10^{-7}$ H/m , $L_{hf}^{external} = 4 \times 10^{-7}$ H/m
- $f_{max} = 1 \times 10^{10}$ Hz , $R_{max} = 2460\ \Omega/m$

→ **RR = 2.07 , LL = 0.351**

Parallel square bars







- R_{dc} = 350 Ω/m , L_{lf}^{total} = 4.8 x 10^{-7} H/m , $L_{hf}^{external}$ = 3.22 x 10^{-7} H/m
- f_{max} = 5 x 10^{10} Hz , R_{max} = 5160 Ω/m

→ RR = 2.36 , LL = 0.448

Compact Equivalent Circuit Models for the Skin Effect

- small R-L ladders (four resistors, three inductors) can provide excellent equivalent circuit for circular conductors
  - good fit from dc to high frequency
  - simple, analytic equations have been established that allow fast calculation of circuit element values for a specified maximum frequency, wire radius, and wire conductivity
- can be used directly to model transmission lines using coupled circular conductors with "weak" proximity effects
  - excellent fit for coaxial cable
  - analytic result for twin lead as a function of wire separation

Compact Equivalent Circuit Models for Skin and Proximity Effects in General Transmission Lines

- for arbitrary cross-section conductors or in the presence of strong proximity effects generalized procedure has been established
  - only one fitting parameter, easily determined via simple error minimization
  - requires knowledge of only R_{dc}, L_{lf}^{total}, $L_{hf}^{external}$, and R_{max} at single high frequency ω_{max}
    - can be determined by calculation or measurement
- excellent fit to detailed calculations for wide range of geometries
  - closely coupled twin lead
  - square to thick, narrow to wide plates
  - also tested for microstrip and strip line, similar excellent agreement
- should provide efficient technique for circuit simulation of lossy transmission lines